A Super Simple Derivation of CAPM

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Portfolio Construction, Expected Return

- Consider a portfolio with w portion invested in an asset i of expected return $r_i := \mathbb{E}(r_{i,t})$ and $1 - w$ portion invested in the market portfolio of expected return $r_m := \mathbb{E}\left(r_{m,t}\right)$.
- The return of this portfolio, denoted by $r_{w,t}$, is a weighted combination of $r_{i,t}$ and $r_{m,t}$.

$$r_{w,t} = w r_{i,t} + (1 - w) r_{m,t} \tag{1}$$

- By the linear property of the expectation operator $\mathbb{E}\left(\ \cdot\ \right)$, the expected return of this portfolio is

$$r_w = w r_i + (1 - w) r_m. \tag{2}$$

Lemma: Variance of $aX + bY$

- Let $\mu_X = \mathbb{E}(X)$, $\mu_Y = \mathbb{E}(Y)$, and $\mu_Z = \mathbb{E}(Z)$.
- Let $Z := aX + bY$.
- $\mu_Z = \mathbb{E}(Z) = a\,\mathbb{E}(X) + b\,\mathbb{E}(Y) = a\mu_X + b\mu_Y$
- The definition of variance of is

$$
\begin{aligned}
\mathbb{V}\left(Z\right) &= \mathbb{E}\left((Z - \mu_Z)^2\right) = \mathbb{E}\left((aX - a\mu_X + bY - b\mu_Y)^2\right) \\
&= \mathbb{E}\left((aX - a\mu_X)^2 + (bY - b\mu_Y)^2 + 2(aX - a\mu_X)(bY - b\mu_Y)\right) \\
&= a^2\,\mathbb{E}\left((X - \mu_X)^2\right) + b^2\,\mathbb{E}\left((Y - \mu_Y)^2\right) \\
&\qquad + 2ab\,\mathbb{E}\left((X - \mu_X)(Y - \mu_Y)\right) \\
&= a^2\,\mathbb{V}(X) + b^2\,\mathbb{V}(Y) + 2ab\,\mathbb{C}(X, Y)
\end{aligned}
$$

Variance of Portfolio's Return

- To (1), apply the variance operator $\mathbb{V}\left(\,\cdot\,\right)$. Using the lemma, we get

$$\mathbb{V}\left(r_{w,t}\right) = w^2\,\mathbb{V}\left(r_{i,t}\right) + (1-w)^2\,\mathbb{V}\left(r_{m,t}\right) + 2w(1-w)\,\mathbb{C}\left(r_{i,t}, r_{m,t}\right).$$

- For convenience, we denote
  - § $\sigma_w^2 := \mathbb{V}\left(r_{w,t}\right)$, $\sigma_i^2 := \mathbb{V}\left(r_{i,t}\right)$ and $\sigma_m^2 = \mathbb{V}\left(r_{m,t}\right)$
  - § The covariance $\sigma_{im} := \mathbb{C}\left(r_{i,t}, r_{m,t}\right)$.

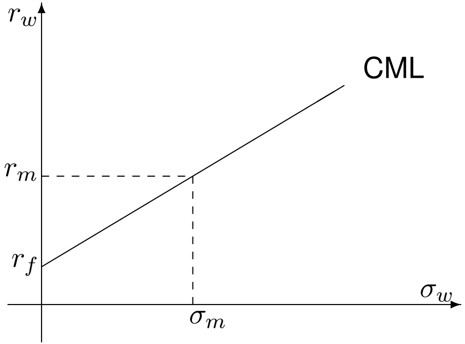
- With these notations, the variance $\mathbb{V}\left(r_{w,t}\right)$ simplifies to

$$\sigma_w^2 = w^2\sigma_i^2 + 2w(1-w)\sigma_{im} + (1-w)^2\sigma_m^2 \tag{3}$$

Slope of CML

1. The slope of the CML is the Sharpe ratio. At $w = 0$, or for σ_m, we have

$$\frac{r_m - r_f}{\sigma_m} = \left.\frac{dr_w}{d\sigma_w}\right|_{w=0}$$



Derivation of Slope

1. It is tedious to compute $\dfrac{dr_w}{d\sigma_w}$ directly.
2. Instead, we have, by chain rule,
$$\frac{dr_w}{d\sigma_w} = \frac{\dfrac{dr_w}{dw}}{\dfrac{d\sigma_w}{dw}}$$
3. From (2), we obtain $\dfrac{dr_w}{dw} = r_i - r_m$
4. From (3), we obtain
$$2\sigma_w \frac{d\sigma_w}{dw} = 2w\sigma_i^2 + 2(1-2w)\sigma_{im}, -2(1-w)\sigma_m^2,$$
equivalently,
$$\frac{d\sigma_w}{dw} = \frac{w\sigma_i^2 + (1-2w)\sigma_{im} - (1-w)\sigma_m^2}{\sigma_w}.$$

Slope at $w = 0$

1. Putting everything together,

$$\frac{dr_w}{d\sigma_w} = \frac{\dfrac{dr_w}{dw}}{\dfrac{d\sigma_w}{dw}} = \frac{r_i - r_m}{\dfrac{w\sigma_i^2 + (1-2w)\sigma_{im} - (1-w)\sigma_m^2}{\sigma_w}}$$

2. At $w = 0$, $\sigma_w = \sigma_m$. Moreover, given that the slope is the Sharpe ratio, we have

$$\frac{r_m - r_f}{\sigma_m} = \frac{r_i - r_m}{\left(\dfrac{\sigma_{im} - \sigma_m^2}{\sigma_m}\right)}$$

$$r_m - r_f = \frac{r_i - r_m}{\left(\dfrac{\sigma_{im} - \sigma_m^2}{\sigma_m^2}\right)} = \frac{r_i - r_m}{\left(\dfrac{\sigma_{im}}{\sigma_m^2} - 1\right)}$$

Slope at $w = 0$ (Cont'd)

3. For any asset i that is not a market portfolio, $\frac{\sigma_{im}}{\sigma_m^2} - 1 \neq 0$. So we multiple it to both sides to obtain

$$(r_m - r_f)\left(\frac{\sigma_{im}}{\sigma_m^2} - 1\right) = r_i - r_m$$

$$\frac{\sigma_{im}}{\sigma_m^2}(r_m - r_f) - (r_m - r_f) = r_i - r_m$$

4. Knowing that $\frac{\sigma_{im}}{\sigma_m^2} = \beta_i$, we write,

$$\beta_i(r_m - r_f) = (r_m - r_f) + r_i - r_m = r_i - r_f$$

5. Hence CAPM ensues:

$$r_i - r_f = \beta_i(r_m - r_f)$$

□